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ON PARTITIONING THE EXPLAINED VARIATION IN A REGRESSION ANALYSIS.

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AN ANALYSIS OF THE NONUNIQUE PORTION OF EXPLAINED VARIATION WHICH CAN BE ATTRIBUTED TO THE REGRESSOR VARIABLES AS A GROUP IN REGRESSION ANALYSIS IS DISCUSSED. THE UNIQUE SUMS OF SQUARES IS PRESENTED AS A BASIS FOR UNDERSTANDING THE PROCEDURE FOR PARTITIONING THE NONUNIQUE PART. A METHOD OF PARTITIONING NONUNIQUE VARIATION IS DEFINED IN TERMS OF A THREE VARIABLE CASE--DEFINITION OF COMMONALITIES, REDUCED FORM OF THE COMMONALITIES, AND PARTITIONING THE SUMS OF SQUARES. TABLES AND AN APPENDIX ARE INCLUDED. (BK)

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NATIONAL CENTER FOR EDUCATIONAL STATISTICS
Division of Operations Analysis

ON PARTITIONING THE EXPLAINED VARIATION IN A REGRESSION ANALYSIS

by

Carl E. Wisler

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The Problem

In exploratory research, regression analysis is often applied to nonexperimental, multivariable data in an attempt to reveal relationships between a dependent variable and a set of regressor variables. Conceptually, the variation in the dependent variable may be separated into three parts, (i) that which can be attributed to the regressor variables individually, (ii) that which can be attributed to the regressor variables as a group and (iii) the residual variation which is unexplained by the regression. Parts (i) and (ii) together comprise the total variation explained by the regression. Since part (i) is composed of the so-called unique sums of squares[1/] it will be referred to as the unique portion of the unexplained variation. The purpose of this note is to analyze the nonunique portion of explained variation (i.e. part (ii)), a subject which has apparently not received much attention heretofore.[2/] The motivation for doing so is the promise of a more complete understanding of the relationship of the dependent variable and the regressor variables. Since the procedure for partitioning the nonunique part may be regarded as an extension of a method for obtaining the unique sums of squares we will begin with a discussion of the latter.

1/ Also called the extra or added sums of squares.

2/ During preparation of this note, two articles were discovered which present essentially the same results as this paper but with another approach and somewhat different motivation. See R.G.Newton and D.J. Spurrell, "A Development of Multiple Regression for the Analysis of Routine Data," <u>Applied Statistics</u>, Vol. 16, No. 1, 1967 and "Examples of the Use of Elements for Clarifying Regression Analysis," <u>Applied Statistics</u>, Vol. 16, No. 2, 1967.

One measure of the variation in the dependent variable is the sum of squares of deviations of the observations from the mean. More specifically, let Y_i denote the i^{th} observation of the dependent variable and let $\bar{Y}$ be the mean of m observations. Then

$$S(Y) = \sum_{i=1}^{m} (Y_i - \bar{Y})^2$$

is the sum of squares of the deviations.

If a regression analysis is carried out using, say, three regressor variables the result is an equation of the form

$$\hat{Y} = b_o + X_1b_1 + X_2b_2 + X_3b_3$$

Given a set of observations, X_{i1}, X_{i2} and X_{i3}, one can then estimate the corresponding value of the dependent variable $\hat{Y}_i$. The amount of variation which is explained by the regression on X_1, X_2 and X_3 is

$$S(X_1X_2X_3) = \sum_{i=1}^{m} (\hat{Y}_i - \bar{Y})^2$$

and the proportion of the total variation which has been accounted for by the regression is

$$R^2(X_1X_2X_3) = \frac{S(X_1X_2X_3)}{S(Y)}$$

which is the square of the multiple correlation coefficient. The residual variation which is unaccounted for by the regression will be denoted as

$$\sum_{i=1}^{m}(Y_i - \hat{Y}_i)^2$$

The procedure for obtaining the unique contribution of, say, X_1 to the sum of squares is to regress Y against X_2 and X_3. This yields $S(X_2X_3)$, the variation accounted for by the variables X_2 and X_3.

The unique sum of squares associated with X_1 is now defined to be

$$\gamma(X_1) = S(X_1X_2X_3) - S(X_2X_3)$$

Thus $\gamma(X_1)$ represents the additional explained sum of squares when X_1 is added to the regression last. (In actual computation $\gamma(X_1)$ can be computed in another way so that it is not necessary to run two complete regressions.)

Using Table 1 we may now look again at our way of classifying the total variation. At this stage of the development the nonunique portion is shown simply as the difference between the total variation and the other two known components. Before proceeding to the analysis some interpretive remarks may be helpful.

Table 1

Classification of the Variation	Sum of Squares
Unique Portion	$\gamma(X_1) + \gamma(X_2) + \gamma(X_3)$
Nonunique Portion	$S(Y) - \sum_i (Y_i-\hat{Y}_i)^2 - \gamma(X_1) - \gamma(X_2) - \gamma(X_3)$
Residual	$\sum_i (Y_i-\hat{Y}_i)^2$
Total	$S(Y)$

The unique contributions of the variables are often calculated because they provide an indication of the relative importance of the several regressor variables in explaining the dependent variable. They are potentially misleading however because they neglect the nonunique portion of explained variation. Looking at the origin of this variation will illuminate the problem.

To examine the nonunique variation more deeply requires the notion of orthogonality among the data vectors, a concept developed in some detail in books such as Goldberger[3/] or Draper and Smith[4/]. Briefly, it can be explained as follows. Let X_i and X_j now be column vectors of observations on the i^{th} and j^{th} variables expressed as deviations from the mean. If there are m observations these will be m-dimensional data vectors. Two such vectors are said to be orthogonal if $X_i'X_j = 0$, that is if the vectors are at right angles to one another in m-space. In general, the cosine of the angle between the vectors is equal to the sample correlation coefficient between the two variables. If a set of data vectors are mutually orthogonal then they are uncorrelated with one another and the unique sums of squares will add up to the total explained sum of squares. In other words, each regressor variable is bringing new information to the regression.

3/ A.S. Goldberger, Econometric Theory, (New York: John Wiley, 1964).
4/ N.R. Draper and H.Smith, Applied Regression Analysis, (New York: John Wiley, 1966).

If the regressors are to some extent redundant, however, we have a departure from orthogonality. When the departure is considerable this condition is known as multicollinearity and is due to high intercorrelations among the regressor variables.[5/] The nonunique portion of explained variation is then nonzero and the problem of determining the contributions of individual variables is inherently ambiguous. The next section, however, describes a way of partitioning the nonunique variation which can be an aid to interpreting the results of regression analyses.

The Three Variable Case

Definition of Commonalities

For ease of exposition, the following discussion of the partitioning procedure will be restricted to three independent variables. The equations required for the n-variable case are given in the Appendix.

First, recall that the unique contribution of X_1 was defined as the sum of squares with all three variables less the sum of squares associated with the regression on X_2 and X_3, i.e.

$$\gamma_1(X_1) = S(X_1X_2X_3) - S(X_2X_3)$$

or in general

(1) $$\gamma_1(X_i) = S(X_iX_jX_k) - S(X_jX_k);$$

Here and subsequently in the three variable case $i = 1,2,3$ and $i \neq j \neq k$.

5/ In the extreme multicollinearity case, a data vector can be written as an exact linear combination of other data vectors. This condition makes regression analysis impossible because matrix inversion cannot be carried out.

This notion has a direct extension in the sense that $S(X_i X_j X_k) - S(X_k)$ is a measure of the effect of adding the variables X_i and X_j to the regression last. We may express this by writing

(2) $S(X_i X_j X_k) - S(X_k) = \gamma_1(X_i) + \gamma_1(X_j) + \gamma_2(X_i X_j)$

where $\gamma_2(X_i X_j)$ is that part of the difference in the sum of squares which may be associated with X_i or X_j. It may be regarded as that part attributable to X_i and X_j in common, or for short, the commonality of $X_i X_j$. In particular since there are two variables involved it will be referred to as a second-order commonality.

Rearranging equation (2) provides us with a definition of second order commonalities, viz.

(3) $\gamma_2(X_i X_j) = S(X_i X_j X_k) - S(X_k) - \gamma_1(X_i) - \gamma_1(X_j)$

The definition is recursive in that the unique sums of squares have been defined by equation (1).

In analogy with equation (2) we may write the total sum of squares attributable to a three variable regression as

(4) $S(X_i X_j X_k) = \gamma_1(X_i) + \gamma_1(X_j) + \gamma_1(X_k) + \gamma_2(X_i X_j) + \gamma_2(X_i X_k)$

$$\gamma_2(X_j X_k) + \gamma_3(X_i X_j X_k).$$

Then rearranging equation (4) we have the definition of the third-order commonality.

(5) $\gamma_3(X_i X_j X_k) = S(X_i X_j X_k) - \gamma_1(X_i) - \gamma_1(X_j) - \gamma_1(X_k) - \gamma_2(X_i X_j)$

$$\gamma_2(X_i X_k) - \gamma_2(X_j X_k)$$

Reduced Form of the Commonalities

Collecting the three equations defining commonalities we have

(1) $\gamma_1(X_i) = S(X_iX_jX_k) - S(X_jX_k)$;

(3) $\gamma_2(X_iX_j) = S(X_iX_jX_k) - S(X_k) - \gamma_1(X_i) - \gamma_1(X_j)$;

(5) $\gamma_3(X_iX_jX_k) = S(X_iX_jX_k) - \gamma_1(X_i) - \gamma_1(X_j) - \gamma_1(X_k) - \gamma_2(X_iX_j)$
$- \gamma_2(X_iX_k) - \gamma_2(X_jX_k)$

The commonality terms on the right side of this set of recursive equations can now be eliminated yielding

(6) $\gamma_1(X_i) = S(X_iX_jX_k) - S(X_jX_k)$

(7) $\gamma_2(X_iX_j) = -S(X_iX_jX_k) + S(X_jX_k) + S(X_iX_k) - S(X_k)$

(8) $\gamma_3(X_iX_jX_k) = S(X_iX_jX_k) - S(X_iX_j) - S(X_iX_k) - S(X_jX_k) + S(X_i)$
$+ S(X_j) + S(X_k)$

The foregoing way of expressing the commonalities will be referred to as the reduced form.

Dividing (6), (7) and (8) by S(Y) gives alternative forms which will be called commonality coefficients.

(9) $U(X_i) = \gamma_1(X_i)/S(Y) = R^2(X_iX_jX_k) - R^2(X_jX_k)$;

(10) $C(X_iX_j) = \gamma_2(X_iX_j)/S(Y) = -R^2(X_iX_jX_k) + R^2(X_jX_k) + R^2(X_iX_k) - R^2(X_k)$;

(11) $C(X_iX_jX_k) = \gamma_3(X_iX_jX_k)/S(Y) = R^2(X_iX_jX_k) - R^2(X_iX_j) - R^2(X_iX_k)$
$- R^2(X_jX_k) + R^2(X_i) + R^2(X_j) + R^2(X_k)$

In this way the part of the total variation associated with a combination of variables can be calculated from the appropriate multiple correlation coefficients.

Partitioning the Sums of Squares

Using the commonality definitions in equations (1), (3) and (5), the sum of squares attributable to a regression can be written in an interesting form. Taking equations (3) and (5) and solving for $S(X_k)$ gives

(12) $S(X_k) = \gamma_1(X_k) + \gamma_2(X_jX_k) + \gamma_2(X_iX_k) + \gamma_3(X_iX_jX_k)$,

solving equations (1) and (5) for $S(X_jX_k)$ gives

(13) $S(X_jX_k) = \gamma_1(X_j) + \gamma_1(X_k) + \gamma_2(X_iX_j) + \gamma_2(X_iX_k) + \gamma_2(X_jX_k)$
$+ \gamma_3(X_iX_jX_k)$

and solving equation (5) for $S(X_iX_jX_k)$ gives

(14) $S(X_iX_jX_k) = \gamma_1(X_i) + \gamma_1(X_j) + \gamma_1(X_k) + \gamma_2(X_iX_j) + \gamma_2(X_iX_k)$
$+ \gamma_2(X_jX_k) + \gamma_3(X_iX_jX_k)$.

In each equation the sum of squares due to the regression is equal to the sum of all unique sums of squares and commonalities associated with the regressor variables. The meaning of the commonalities is now clear in terms of equation (12). For example, $\gamma_2(X_jX_k)$ is that part of the sum of squares which is common to $S(X_j)$ and $S(X_k)$ and no other; $\gamma_3(X_iX_jX_k)$ is common to $S(X_i)$, $S(X_j)$ and $S(X_k)$ and no other; etc.

Group of Variables

The equations presented can easily be extended to apply to groups of regressor variables. For example, let F represent the set of variables X_1, X_2, X_3 ; let G represent the set X_4, X_5 and let H represent X_6, X_7 and X_8. Then

$$R^2(F) = R^2(X_1X_2X_3)$$
$$R^2(G) = R^2(X_4X_5)$$
$$R^2(H) = R^2(X_6X_7X_8)$$
$$R^2(FG) = R^2(X_1X_2X_3X_4X_5)$$

and so on.

The commonality coefficient for two sets of variables, say F and G, is then

$$C(FG) = R^2(FGH) - R^2(H) - C(F) - C(G)$$

by extension of equation (3) and other commonality coefficients follow directly. Note that interpretation of the coefficients must be modified slightly now; for example, a unique contribution may refer to the variation explained by a group of variables rather than a single one. The grouping device will be used in the example which follows.

An Example

To illustrate the use of commonalities we shall use some data collected as part of the Educational Opportunities Survey (EOS).[6/]

6/ Coleman, J.S., et al., Equality of Educational Opportunity. U.S. Department of Health, Education and Welfare; National Center for Educational Statistics, (OE-38001). Washington, D.C. 1966, U.S. Government Printing Office Catalog No. FS 5.238:38001.

The dependent variable is an index of achievement developed from the EOS data.[7/] A large number of regressor variables were separated into four groups as follows: student background variables (B), teacher variables (T), school program variables (P) and school facilities variables (F). Since this note is primarily concerned with methodology we need not go deeper into the nature of these variables; readers more interested in the content should refer to Mayeske, et al.[8/]

Table 2 displays the set of commonality coefficients for the data. For second and higher orders a table entry is made for each variable with which a coefficient is associated.

7/ Mayeske, G.W. and F.D.Weinfeld, Factor Analysis of Achievement Measures From the Educational Opportunities Survey, Division of Operations Analysis, Technical Note No. 21, January 18, 1967.

8/ Mayeske, G.W., F.D.Weinfeld, A.E.Beaton, Jr., and J.M.Proshek, Correlational and Regression Analysis of Differences Between the Achievement Levels of Ninth Grade Schools from the Educational Opportunities Survey, Division of Operations Analysis, Unpublished manuscript.

Table 2

Commonality Coefficients

Commonality Coefficients		Sets of Regressor Variables B	T	P	F
First Order	U(B)	.1061			
	U(T)		.0167		
	U(P)			.0125	
	U(F)				.0038
Second Order	C(BT)	.4891	.4891		
	C(BP)	.0137		.0137	
	C(BF)	.0004			.0004
	C(TP)		.0066	.0066	
	C(TF)		-.0009		-.0009
	C(PF)			.0050	.0050
Third Order	C(BTP)	.1197	.1197	.1197	
	C(BTF)	.0304	.0304		.0304
	C(BPF)	.0052		.0052	.0052
	C(TPF)		.0018	.0018	.0018
Fourth Order	C(BTPF)	.0561	.0561	.0561	.0561
R^2 for a Single Set of Variables		.8207	.7195	.2206	.1018

The last row of the table sums the coefficients for each column thus giving the square of the multiple correlation coefficient for each individual variable as suggested by equation (12). The sum of all coefficients is R^2(BTPF) = .8662.

This set of results shows how knowledge of the higher order commonalities can provide additional insight into the relationship among the variables. Looking at only the unique contributions of the variables suggests that the student background variables with U(B) = .1061 outweigh the others in explaining achievement. However, the second order coefficient between student background and teacher variables is .4891, the largest of all coefficients. This means that when these two sets of variables are added to the regression last, the reduction in unexplained variance is substantially more than can be attributed to each set individually on the basis of first order coefficients. Though we are not able to further separate this joint contribution we are at least warned that the effect of the teacher variables may be much greater than was indicated by the first order coefficients.

The joint effects also carry through to higher order commonality coefficients. Thus C(BTP) is .1197, the second largest coefficient, and C(BTPF) is .0561 the fourth largest. Consequently, one has good reason to guard against rejecting school variables in general and especially teacher variables as determinants of achievement. Resolution of the ambiguity is another matter however for it is generally agreed that correcting the effects of multicollinearity requires the acquisition of new data.[9]

The perhaps unexpected result that commonalities can be negative is evident from Table 2. Re-examination of their development shows that, unlike the unique sums of squares, they are not constrained from being negative. Exactly how negative commonalities should be interpreted is, at this time, an open question though the previously cited reference by Newton and Spurrell offers a geometric explanation for their occurence.

Acknowledgement

The notion of commonalities was suggested by A. M. Mood in a personal communication in which he developed the reduced form and the symbolic form of the Appendix. The author has also benefited from discussions with D. S. Stoller, F. D. Weinfeld, G. W. Mayeske and J. M. Proshek.

9/ See for example, J. Johnston, Econometric Methods, (New York: McGraw-Hill, 1963, p.207).

APPENDIX

Partitioning in the n-variable Case

Writing out expressions for commonalities in the n-variable case requires some set notation to keep track of the variables. We will use the following:

V^n	The set of all variables $X_1, X_2, ---, X_n$.
V^j	A subset of V^n containing j members.
$\bar{V}^j$	The complement of V^j.
V_i^j	A subset of V^j containing i members.
$\{V_i^j\}$	The set of all possible V_i^j for a given V^j; The number of members in the set is $\binom{j}{i} = \frac{j!}{i!(j-i)!}$
$\bar{V}^j V_i^j$	The union of the sets $\bar{V}^j$ and V_i^j.

For the three variable case the set memberships are given in Table A1. The commonality definitions may be written as

$$\gamma_1(V^1) = S(V^n) - S(\bar{V}^1)$$

$$\gamma_2(V^2) = S(V^n) - S(\bar{V}^2) - \sum_{\{V_1^2\}} \gamma_1(V_1^2)$$

$$\gamma_3(V^3) = S(V^n) - S(\bar{V}^3) - \sum_{\{V_2^3\}} \gamma_2(V_2^3) - \sum_{\{V_1^3\}} \gamma_1(V_1^3)$$

$$\vdots$$

$$\gamma_k(V^k) = S(V^n) - S(\bar{V}^k) - \sum_{\{V_{k-1}^k\}} \gamma_{k-1}(V_{k-1}^k) - \sum_{\{V_{k-2}^k\}} \gamma_{k-2}(V_{k-2}^k) - \cdots - \sum_{\{V_1^k\}} \gamma_1(V_1^k)$$

$$\vdots$$

$$\gamma_n(V^n) = S(V^n) - \sum_{\{V_{n-1}^n\}} \gamma_{n-1}(V_{n-1}^n) - \sum_{\{V_{n-2}^n\}} \gamma_{n-2}(V_{n-2}^n) - \cdots - \sum_{\{V_1^n\}} \gamma_1(V_1^n)$$

<table>
<tr><th>Set Symbol</th><th colspan="6">Set Memberships</th></tr>
<tr><td>V^1
$\bar{V}^1$</td><td colspan="2">X_1
X_2X_3</td><td colspan="2">X_2
X_1X_3</td><td colspan="2">X_3
X_2X_3</td></tr>
<tr><td>V^2
$\bar{V}^2$</td><td colspan="2">X_1X_2
X_3</td><td colspan="2">X_1X_3
X_2</td><td colspan="2">X_2X_3
X_1</td></tr>
<tr><td>V_1^2
$\bar{V}^2V_1^2$</td><td>X_1
X_1X_3</td><td>X_2
X_2X_3</td><td>X_1
X_1X_2</td><td>X_3
X_2X_3</td><td>X_2
X_1X_2</td><td>X_3
X_1X_3</td></tr>
<tr><td>V^3
$\bar{V}^3$</td><td colspan="6">$X_1X_2X_3$
empty</td></tr>
<tr><td>V_2^3
$\bar{V}^3V_2^3$
$\bar{V}^3V_1^3$</td><td colspan="2">X_1X_2
X_1X_2
X_1</td><td colspan="2">X_1X_3
X_1X_3
X_2</td><td colspan="2">X_2X_3
X_2X_3
X_3</td></tr>
</table>

Table A1

The number of k^{th} order commonalities is $n!/k!(n-k)!$ and consequently the total number of commonalities is 2^n-1.

The reduced forms may be written as

$$\gamma_1(V^1) = S(V^n) - S(\bar{V}^1)$$

$$\gamma_2(V^2) = -S(V^n) + \sum_{\{V_1^2\}} S(\bar{V}^2 V_1^2) - S(\bar{V}^2)$$

$$\gamma_3(V^3) = S(V^n) - \sum_{\{V_2^3\}} S(\bar{V}^3 V_2^3) + \sum_{\{V_1^3\}} S(\bar{V}^3 V_1^3) - S(\bar{V}^3)$$

$$\vdots$$

$$\gamma_k(V^k) = (-1)^{k+1}\Big[S(V^n) - \sum_{\{V_{k-1}^k\}} S(\bar{V}^k V_{k-1}^k) + \sum_{\{V_{k-2}^k\}} S(\bar{V}^k V_{k-2}^k) - \ldots (-1)^{k+1} \sum_{\{V_1^k\}} S(\bar{V}^k V_1^k)\Big] - S(\bar{V}^k)$$

$$\vdots$$

$$\gamma_n(V^n) = (-1)^{n+1}\Big[S(V^n) - \sum_{\{V_{n-1}^n\}} S(V_{n-1}^n) + \sum_{\{V_{n-2}^n\}} S(V_{n-2}^n) - \ldots (-1)^{n+1} \sum_{\{V_1^n\}} S(V_1^n)\Big]$$

An alternative way of keeping track of the variables for the reduced form is by a symbolic form for the argument. We give an example first and then write the general form. A second order commonality in the three variable case may be written as

$$\gamma_2(V^2) = \gamma_2(X_iX_j) = S\left[-(1-X_i)(1-X_j)X_k\right]$$

The meaning of the symbolic expression on the right is that the product in the brackets is first multiplied out and then the absolute value of each term becomes an argument for a sum of squares with the sign of the term carrying over as the sign on the sum of squares. That is,

$$\begin{aligned}\gamma_2(X_iX_j) &= S\left[-(1-X_i)(1-X_j)X_k\right] \\ &= S\left[-X_k+X_jX_k+X_iX_k-X_iX_jX_k\right]\end{aligned}$$

and upon converting from the symbolic form we have

(A1) $\gamma_2(X_iX_j) = -S(X_iX_jX_k) + S(X_jX_k) + S(X_iX_k) - S(X_k)$.

Equation (A1) is the same as equation (7).

Using the symbolic form we may write the commonalities which compose $S(X_1)$ as

$$\gamma_1(X_1) = S\left[-(1-X_1)X_2X_3\ldots X_n\right]$$

$$\gamma_2(X_1X_2) = S\left[-(1-X_1)(1-X_2)X_3X_4\ldots X_n\right]$$

$$\vdots$$

$$\gamma_{n-1}(X_1X_2\ldots X_{n-1}) = S\left[-(1-X_1)(1-X_2)\ldots(1-X_{n-1})X_n\right]$$

$$\gamma_n(X_1X_2\ldots X_n) = S\left[1-(1-X_1)(1-X_2)\ldots(1-X_n)\right]$$

The other commonalities have analogous forms.